GR

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July 21, 2021

Contents

Matter tells SpaceTime where to curve and SpaceTime tells Matter how to move.

1 Contravarient Transformation

Consider a cartesian coordinate system of d dimensions $(x^1, x^2, ..., x^d)$. Also, consider another curvilinear coordinate system $y^1, ...$ where $y's$ is a function of $x's$

For a very small displacement, transformation equation is,

$$dy^n = \frac{\partial y^n}{\partial x^m} dx^m \tag{1.1}$$

For a Vector A^r,

$$A^n(y) = \frac{\partial y^n}{\partial x^m} A^m(x) \tag{1.2}$$

Similarly for a Tensor T^{rs} such that $T^{rs} = A^r B^s$ the transformation equation is given by,

$$T^{mn}(y) = \frac{\partial y^m}{\partial x^r} \frac{\partial y^n}{\partial x^s} A^r B^s = \frac{\partial y^m}{\partial x^r} \frac{\partial y^n}{\partial x^s} T^{rs}(x) \tag{1.3}$$

Any Tensor which transforms as (1.3) is called a **Contravarient Tensor**.

1.1 Insights

Contravarient Transformation is similar to transformation of a small differential element.

$$dy^m = \frac{\partial y^m}{\partial x^n} dx^n$$
$$V^m(y) = \frac{\partial y^m}{\partial x^n} V^n(x)$$

2 Covarient Transformation

Consider a cartesian coordinate system of d dimensions $(x^1, x^2, ..., x^d)$. Also, consider another curvilinear coordinate system $y^1, ...$ where $y's$ is a function of $x's$

$$A_n(y) = \frac{\partial x^m}{\partial y^n} A_m(x) \tag{2.1}$$

$$T_{mn}(y) = \frac{\partial x^m}{\partial y^r} \frac{\partial x^n}{\partial y^s} T_{rs}(x) \tag{2.2}$$

Any Tensor which transforms as (2.2) is called a **Covarient Tensor**.

2.1 Insights

Covarient Transformation is similar to transformation of a gradient.

$$\frac{\partial \phi}{\partial y^m} = \frac{\partial x^n}{\partial y^m} \frac{\partial \phi}{\partial x^n}$$
$$V_m(y) = \frac{\partial x^n}{\partial y^m} V_n(x)$$

3 Riemanian and pseudo-Riemanian Geometry

$$s = \text{No. of +1 in the Metric} t = \text{No. of -1 in the Metric}$$

When $t = 0$, the Geometry is Riemanian.

When $t = 1$, the Geometry is pseudo-Riemanian.

4 Metic Tensor

Consider a cartesian coordinate system of d dimensions $(x^1, x^2, ..., x^d)$. Also, consider another curvilinear coordinate system $y^1, ...$ where $y's$ is a function of $x's$

Consider a vector of length ds, then

$$ds^2 = (dx^1)^2 + (dx^2)^2 + \ \dots \tag{4.1}$$

$$ds^2 = \Sigma_m (dx^m)^2 \tag{4.2}$$

Or,

$$ds^2 = \Sigma_m dx^m dx^m \tag{4.3}$$

But, (4.3) violates or rule that the repeated indices(dummy) must be one in superscript and the other in subscript, so we define a function called **Kronecker Delta**.

$$\delta_{mn} = \begin{cases} 1 & \text{iff } m = n \\ 0 & \text{iff } m \neq n \end{cases} \tag{4.4}$$

Therefore, (4.3) can be rewritten as,

$$ds^2 \ = \ \delta_{mn} \ dx^m \ dx^n \tag{4.5}$$

In Equation 4.5, the Kronecker Delta is the **Metric for Cartesian Space**.

Now, for metric of a curvilinear space,

$$ds^2 = \delta_{mn} \frac{\partial x^m}{\partial y^r} \frac{\partial x^n}{\partial y^s} \, dy^r \, dy^s \tag{4.6}$$

$$ds^2 = g_{mn} dy^m dy^n \tag{4.7}$$

where $g_{mn} = \delta_{mn} \frac{\partial x^m}{\partial y^r} \frac{\partial x^n}{\partial y^s}$ called the **Metric**.

4.1 Transformation of the Metric Tensor

$$\begin{aligned} ds^2 \ &= \ g_{mn}(x) \ dx^m \ dx^n \\ ds^2 \ &= \ g_{mn}(x) \ \frac{\partial x^m}{\partial y^r} \ \frac{\partial x^n}{\partial y^s} \ dy^r \ dy^s \end{aligned} \tag{4.8}$$

Hence,

$$g_{mn}(y) \;=\; g_{mn}(x)\,\frac{\partial x^m}{\partial y^r}\,\frac{\partial x^n}{\partial y^s}$$

which is similar to Equation 2.2, Therefore g_{mn} is a Contravarient Tensor.

4.2 About the Metric Tensor

The Metric Tensor defines a small infinitisimal length in a space, has d^2 elements.

Metric Tensor for Cartesian Coordinates with drei dimensions is

$$g_{mn} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Suppose, the cross-term(s), g_{mn} where $m \neq n$, isn't zero, then it means that the x^m and x^n aren't independent and the distance equation would include $dx^m dx^n$ term.

{Suppose, the cross-term(s), g_{12}, isn't zero, then it means that the x^1 and x^2 aren't independent and the distance equation would include $dx^1 dx^2$ term.}

4.3 Inverse of a Matrix Tensor

Inverse of any quantity is defined as, if a matrix is multiplied by its inverse then it should results into the Identity Matrix.

Identity Matrix is the Kronecker Delta, δ^m_n

$$\delta^m_n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$(g^{-1})^{mr}\, g_{rs} \;=\; \delta^m_s$$

Here $(g^{-1})^{mr}$ is the **Contravarient Representation of the Metric Tensor**, and can be simply denoted as g^{mn}.

$$g^{mr}\, g_{rs} \;=\; \delta^m_s \tag{4.9}$$

Symmetry of the Metric Tensor

For

$$g_{mn} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

$$ds^2 \;=\; g_{mn}\, dx^m\, dx^n$$

$$ds^2 \;=\; g_{11}\, dx^1 dx^1 + (g_{12} + g_{21})\, dx^1\, dx^2 + g_{22}\, dx^2\, dx^2$$

Interestingly, Susskind said that Since the Metric is supposed to be invertable then it should be symmetric. Check This.......

Symm Proof

4.4 Covarient ↔ Contravarient

Interestingly, the same quantity can be represented by covarient tensors and contravarient tensors, they can be interconverted using the Metric Tensor.

$$V_m = g_{mn} V^n \tag{4.10}$$
$$V^m = g^{mn} V_m \tag{4.11}$$

If two Tensors are equal in one coordinate system then they must also be equal in other coordinate systems.

$$T_{mn}(x) = W_{mn}(x)$$

The equality of tensors is a **Geometric Fact** so it doesn't depend on the coordinates.

Surprisingly, the individual components of tensors needn't be independent of the coordinate system, I mean, they depend upon the coordinate system.

5 Flatness

Any space is termed as **Flat** when there exists a coordinate transformation which makes the metric tensor resemble the metric of the cartesian coordiantes.

For Example

1. Cylinders are **Flat** since they can be unfolded into a flat sheet.
2. Cones are **Flat** except at the top point.
3. Spheres are **Not Flat**.

6 Metric Tensor For Polar Coordinates

$$ds^2 = \delta_{mn} dx^m dx^n$$

Transformation Eqs.,

$$x = rcos(\theta)$$
$$y = rsin(\theta)$$

Therefore, dx and dy are,

$$dx = cos(\theta)dr - rsin(\theta)d\theta$$
$$dy = sin(\theta)dr + rcos(\theta)d\theta$$

The infinitisimal distance, ds^2 is given by,

$$ds^2 = dr^2 + r^2 d\theta^2$$

Hence, the Metric Tensor in Polar Coordinates is,

$$g_{mn} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

It is interesting to note that the Metric Tensor actually depends upon the Radius r.

Contravarient Representation of the Metric Tensor,

$$g^{mn} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{r^2} \end{pmatrix}$$

7 Minkowski Space

Minkowski Space is a combination of three-dimentional space coordinates and one-dimentional time coordinates.

The quantity which remains invarient under Lorentz Transformation in Minkowski Space is the proper time, or the distance.

For $c = 1$

$$ds^2 = d\tau^2 = dt^2 - dx^2 - dy^2 - dz^2$$
$$d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu$$

$$where\ g_{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$d\tau^2 = \eta_{\mu\nu} dx^\mu dx^\nu$$

8 Tensor! Not a Tensor!

8.1 Property

Suppose

$$T_{mn} = W_{mn} \qquad T_{mn} - W_{mn} = 0 \tag{8.1}$$

In a coordinate system, then also Equation 8.1 will hold in other coordinate systems, although the individual components are prone to change.

8.2 Tensor Ordinary and Covarient Derivative

The Derivative of Tensor is not a Tensor!

Let's Suppose $T_{mn}(x) = \frac{\partial V_m}{\partial x^n}$ is a tensor, then it should behave like a tensor and should trasform w.r.t the equation,

$$T_{mn}(y) = \frac{\partial x^r}{\partial y^m} \frac{\partial x^s}{\partial y^n} T_{rs}(x) \tag{8.2}$$

$$\begin{aligned} RHS &= \frac{\partial x^r}{\partial y^m} \frac{\partial x^s}{\partial y^n} T_{rs}(x) \\ &= \frac{\partial x^r}{\partial y^m} \frac{\partial x^s}{\partial y^n} \frac{\partial V_r}{\partial x^s} \qquad \text{(From the def of } T_{rs}) \\ &= \frac{\partial x^r}{\partial y^m} \frac{\partial V_r}{\partial yn} \end{aligned} \tag{8.3}$$

$$\begin{aligned} LHS &= T_{mn}(y) \\ &= \frac{\partial V_m(y)}{\partial y^n} \qquad \text{(From the def of } T_{rs}) \\ &= \frac{\partial}{\partial y^n} \left(\frac{\partial x^r}{\partial y^m} V_r \right) \\ &= \frac{\partial}{\partial y^n} \frac{\partial x^r}{\partial y^m} V_r + \frac{\partial x^r}{\partial y^m} \frac{\partial V_r}{\partial y^n} \end{aligned} \tag{8.4}$$

$$\therefore \quad LHS \neq RHS$$

The Equation 8.3 and Equation 8.4 aren't equal when $\frac{\partial}{\partial y^n}\frac{\partial x^r}{\partial y^m} \neq 0$, i.e. for curvilinear coordinate systems.

Hence, our assumption that T_{mn} is a Tensor is wrong.

Therefore, **Ordinary Derivative of a Tensor is not a Tensor.**

Actually,

$$T_{mn}(y) \;=\; \frac{\partial}{\partial y^n}\frac{\partial x^r}{\partial y^m}\, V_r(x) \;+\; \frac{\partial V_m(y)}{\partial y^n} \tag{8.5}$$

$$\text{Let, } \Gamma^r_{nm} \;=\; \frac{\partial}{\partial y^n}\frac{\partial x^r}{\partial y^m}$$

$$T_{mn}(y) = \Gamma^r_{nm} V_r(x) + \frac{\partial V_m(y)}{\partial y^n} \tag{8.6}$$

And, now we can define the ***Covarient Derivative*** from Equation 8.6 as,

$$\nabla_n = \Gamma^r_{nm} + \frac{\partial}{\partial y^n} \tag{8.7}$$

where, Γ^r_{nm} represents **Christoffel Symbol**.

Hence, the Equation 8.6 becomes,

$$T_{mn}(y) = \nabla_n V_m \tag{8.8}$$

Interesting: It is interesting to note here that the covarient derivative of a scaler is same as its ordinary derivative.

Rectify

8.3 Covarient Derivative of other Sorts of Tensors!

Let us assume the covarient derivative of other sorts of tensors is given by,

$$\nabla_\mu w_\lambda = \partial_\mu w_\lambda + \tilde{\Gamma}^\sigma_{\mu\lambda} w_\sigma \tag{8.9}$$

Assumptions:

- $\nabla_\mu(\phi) = \partial_\mu \phi$ Covarient Derivative of a scaler is same as its oridinary derivative.
- $\nabla_\mu T^\lambda_{\lambda\nu} = (\nabla T)_\mu{}^\lambda{}_{\lambda\nu}$ Contraction Commutes.

$$\therefore \quad \nabla_\mu(w_\lambda V^\lambda) = V^\lambda \partial_\mu(w_\lambda) + w_\lambda \partial_\mu(V^\lambda) + V^\lambda \tilde{\Gamma}^\sigma{}_{\mu\lambda} w_\sigma + w_\lambda \Gamma^\lambda{}_{\mu\rho} V^\rho \tag{8.10}$$

Since Equation 8.10 is covarient derivative of a scaler, it should result into only partial derivative of the scaler,

$$\nabla_\mu(w_\lambda V^\lambda) = V^\lambda \partial_\mu(w_\lambda) + w_\lambda \partial_\mu(V^\lambda) \tag{8.11}$$

$$\therefore \quad V^\lambda \tilde{\Gamma}^\sigma{}_{\mu\lambda} w_\sigma + w_\lambda \Gamma^\lambda{}_{\mu\rho} V^\rho = 0 \tag{8.12}$$

$$V^{\lambda}\tilde{\Gamma}^{\sigma}{}_{\mu\lambda}w_{\sigma} = -w_{\lambda}\Gamma^{\lambda}{}_{\mu\rho}V^{\rho} \tag{8.13}$$

But, w_{λ} and V^{λ} are arbitary and but changing the dummy index,

$$\tilde{\Gamma}^{\sigma}{}_{\mu\lambda} = -\Gamma^{\sigma}{}_{\mu\lambda} \tag{8.14}$$

Hence from Equation 8.9,

$$\nabla_{\mu}w_{\lambda} = \partial_{\mu}w_{\lambda} - \Gamma^{\rho}{}_{\mu\lambda}w_{\rho} \tag{8.15}$$

$$\nabla_{\mu} = \partial_{\mu} - \Gamma^{\rho}{}_{\mu\lambda} \tag{8.16}$$

8.4 Covarient Derivative of a Tensor

$$\nabla_{p}T_{mn} = \frac{\partial T_{mn}}{\partial y^{p}} - \Gamma^{r}{}_{pm}T_{rn} - \Gamma^{r}{}_{pn}T_{mr} \tag{8.17}$$

8.5 Tangent Vector

For a field ϕ, the tangent vector is $\frac{d\phi}{ds} = \frac{\partial\phi}{\partial x^{n}}\frac{dx^{n}}{ds}$

Similarly, the tangent vector for a Vector V_m is

$$\frac{dV_m}{ds} = \frac{\partial V_m}{\partial x^{n}}\frac{dx^{n}}{ds}$$

But, the partial derivative of a Tensor doesn't results into a tensor, so we replace the partial derivative with covarient derivative.

$$\frac{dV_m}{ds} = \nabla_{p}V_m\frac{dx^{n}}{ds}$$

On, expanding using Equation 8.17,

$$\frac{dV_m}{ds} = \left\{\frac{\partial V_m}{\partial y^{p}} - \Gamma^{r}_{pm}V_r\right\}\frac{dx^{n}}{ds}$$

Interesting! The Covarient Derivative is equal to the Ordinary Derivative for a Flat Cartesian Space.

More Interesting! The Covarient Derivative is *Zero* for a vector which doesn't vary. For Ex, suppose the ∇_p is zero for a Tangent Vector then the curve must be the straightest curve possible in the space (Underlying ref: Geodesics), for a Cartesian Geometry the curve will be a straight line.

9 Covarient Derivative of The Metric Tensor

$$\nabla_{p}g_{mn} = \frac{\partial g_{mn}}{\partial y^{p}} - \Gamma^{r}_{pm}g_{rn} - \Gamma^{r}_{pn}g_{mr} \tag{9.1}$$

For a flat Geometry, Equation 9.1 is **Zero**.

Proof,

The Christoffel Symbols for a flat geoemtry are,

$$\Gamma^{\gamma}{}_{\alpha\beta} = \frac{1}{2} g^{\gamma\sigma} \left\{ \frac{\partial g_{\sigma\beta}}{\partial y^p} + \frac{\partial g_{\sigma\alpha}}{\partial y^m} - \frac{\partial g_{\alpha\beta}}{\partial y^\gamma} \right\}$$

RHS of Equation 9.1

$$RHS = \frac{\partial g_m n}{\partial y^p} - \frac{1}{2} \left\{ g^{rd} \{ \frac{\partial g_d m}{\partial y^p} + \frac{\partial g_{dp}}{\partial y^m} - \frac{\partial g_{pm}}{\partial y^r} \} g_{rn} + g^{r'd'} \{ \frac{\partial g_{d'n}}{\partial y^p} + \frac{\partial g_{d'p}}{\partial y^n} - \frac{\partial g_{pn}}{\partial y^{r'}} \} g_{mr} \right\} \tag{9.2}$$

Since, the Geometry is flat, it implies that $d = r$ and $d' = r'$,

$$RHS = \frac{\partial g_m n}{\partial y^p} - \frac{1}{2} \left\{ \delta^r{}_n \{ \frac{\partial g_d m}{\partial y^p} + \frac{\partial g_{dp}}{\partial y^m} - \frac{\partial g_{pm}}{\partial y^r} \} + \delta^{r'}{}_m \{ \frac{\partial g_{d'n}}{\partial y^p} + \frac{\partial g_{d'p}}{\partial y^n} - \frac{\partial g_{pn}}{\partial y^{r'}} \} \right\} \tag{9.3}$$

From the properties of the metric tensor $g^{rr} g_{rn} = \delta^r{}_n$,

Therefore, $r = n$ and $r' = m$,

$$RHS = \frac{\partial g_m n}{\partial y^p} - \frac{1}{2} \left\{ \frac{\partial g_{nm}}{\partial y^p} + \frac{\partial g_{np}}{\partial y^m} - \frac{\partial g_{pm}}{\partial y^n} + \frac{\partial g_{mn}}{\partial y^p} + \frac{\partial g_{mp}}{\partial y^n} - \frac{\partial g_{pn}}{\partial y^m} \right\} \tag{9.4}$$

From the properties of the metric tensor $g_{mn} = g_{nm}$,

Hence,

$$\nabla_p g_{mn} = RHS = 0$$

10 Custom Geometry

$$Consider\ g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a(t) & 0 & 0 \\ 0 & 0 & a(t) & 0 \\ 0 & 0 & 0 & a(t) \end{pmatrix} \{where\ a(t)\ represents\ a\ time\ dependent\ function.\}$$

This metric can represent a time dependent space-time.

For Example,

- For Coordinates, $ds^2 = dr^2 + (rsin(\phi))^2 d\theta^2$ This will represent a circle at angle ϕ.
- For Coordinates, $ds^2 = dr^2 + e^{2r} d\theta^2$ This will represent a horn like space-time.

11 Kronecker Delta Function

Kronecker Delta Function is Delta function in two dimensions.

11.1 Delta Funciton

$$\delta(x-a) = \begin{cases} 1 & \text{iff } x = a \\ 0 & \text{iff } x \neq a \end{cases} \tag{11.1}$$

The Delta Function can also be thought as a slope to the Heaviside Function.

Heaviside Function is Basically,

$$H(x-a) = \begin{cases} 1 & \text{iff } x \geq a \\ 0 & \text{iff } x < a \end{cases} \tag{11.2}$$

11.2 Kronecker Delta

$$\delta_{ij} = \begin{cases} 1 & \text{iff } i = j \\ 0 & \text{iff } i \neq j \end{cases} \tag{11.3}$$

11.2.1 Tensor! Not a Tensor!

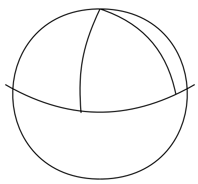
$$Form\ \delta_{mn}$$

$$\delta_{mn} = \frac{\partial x^r}{\partial y^m}\frac{\partial x^s}{\partial y^n}\delta_{rs}$$
$$\delta_{mn} = \frac{\partial x^r}{\partial y^m}\frac{\partial x^r}{\partial y^n}$$
$$Not\ a\ Tensor!$$

$$Form\ \delta^m_n$$

$$\delta^m_n = \frac{\partial y^m}{\partial x^r}\frac{\partial x^s}{\partial y^n}\delta^r_s\delta^m_n = \frac{\partial y^m}{\partial x^s}\frac{\partial x^s}{\partial y^n}\delta^m_n = \frac{\partial y^m}{\partial y^n}\delta^m_n(y) = \delta^m_n(x)$$

Kronecker Delta of the form δ^i_j is a tensor since it transforms appropiately,



12 Geodesic

Geodesic is a curve where the Covarient Derivative of the tangent vector along the curve is zero.

$$\text{Let,}\quad V^\mu = \frac{dx^\mu}{ds} \tag{12.1}$$

$$\text{Since}\quad \frac{dV^\mu}{ds} = \nabla_\lambda V^\mu \frac{dx^\lambda}{ds}$$
$$\therefore\quad \frac{dV^\mu}{ds} = \frac{\partial V^\mu}{\partial x^\lambda}\frac{dx^\lambda}{ds} + \Gamma^\mu{}_{\lambda\sigma}V^\sigma\frac{dx^\lambda}{ds}$$

$$\text{Using Equation 12.1,} \quad \frac{dV^\mu}{ds} = \frac{\partial}{\partial x^\lambda}\frac{dx^\mu}{ds}\frac{dx^\lambda}{ds} + \Gamma^\mu{}_{\lambda\sigma}\frac{dx^\sigma}{ds}\frac{dx^\lambda}{ds}$$

$$\frac{dV^\mu}{ds} = \frac{d^2x^\mu}{ds^2} + \Gamma^\mu{}_{\lambda\sigma}\frac{dx^\sigma}{ds}\frac{dx^\lambda}{ds} \tag{12.2}$$

For, Equation 12.2 is Zero, the curve is the Geodesic.

12.1 Weak Graviational Field

For Weak Graviational Field and everything moves slowly, ofcourse c=1(HaHa!)

$$g_{\mu\nu} = \eta_{\mu\nu} + \text{Small Corrections}\dots \tag{12.3}$$

In this case, $x^0 \sim \tau \sim t$

$$\therefore, \quad \frac{dx^0}{d\tau} = 1 + \text{Small Correction}\dots \tag{12.4}$$

$$\text{Since, the velocity is very small,} \frac{dx^\alpha}{d\tau} \sim 0$$

$$\text{Using, Equation 12.2 and assuming Geodesic} \quad \frac{dV^\mu}{ds} = \frac{d^2x^\mu}{ds^2} + \Gamma^\mu{}_{00}\frac{dx^0}{ds}\frac{dx^0}{ds} = 0$$

$$\text{Since, Equation 12.4,} \quad \frac{d^2x^\mu}{ds^2} = -\Gamma^\mu{}_{00} \tag{12.5}$$

$$\text{Since,} \quad \Gamma^\gamma{}_{\alpha\beta} = \frac{1}{2}\, g^{\gamma\kappa}\Big\{\partial_\alpha g_{\kappa\beta} + \partial_\beta g_{\kappa\alpha} - \partial_\kappa g_{\alpha\beta}\Big\} \tag{12.6}$$

$$\therefore \quad \Gamma^\mu{}_{00} = \frac{1}{2}\,\eta^{\mu\mu}\Big\{\partial_0 g_{\mu 0} + \partial_0 g_{\mu 0} - \partial_\mu g_{00}\Big\} \tag{12.7}$$

$$\text{Let,} \quad \eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\text{Therefore, Equation 12.5 becomes,} \quad \frac{d^2x^\mu}{ds^2} = -(-)\frac{1}{2}\Big\{\partial_0 g_{\mu 0} + \partial_0 g_{\mu 0} - \partial_\mu g_{00}\Big\}$$

$$\frac{d^2x^\mu}{ds^2} = \frac{1}{2}\Big\{\partial_0 g_{\mu 0} + \partial_0 g_{\mu 0} - \partial_\mu g_{00}\Big\} \tag{12.8}$$

$$\text{Since, the time dependence is very small,} \quad \frac{d^2x^\mu}{ds^2} = -\frac{1}{2}\,\partial_\mu g_{00} \tag{12.9}$$

$$\text{Using the Newton's Equivalent of Acceleration,} \quad \frac{d^2x}{ds^2} = -\frac{d\phi}{dx}$$

$$\therefore \quad \frac{d\phi}{dx} = \frac{1}{2}\frac{dg_{00}}{dx} \tag{12.10}$$

$$\text{On Integration,} \quad g_{00} = 2\,\phi + C \tag{12.11}$$

13 Curvature: Riemann Tensor, Ricci Tensor

13.1 Commutators

$$[\, A, B \,] \;=\; AB \;-\; BA \tag{13.1}$$

Theorem:

$$[\, \partial_x , f(x) \,] \, V(x) \;=\; \{\partial_x f(x) - f(x)\partial_x\} \; V(x)$$

$$[\, \partial_x , f(x) \,] \, V(x) \;=\; \partial_x f(x) \, V(x) - f(x)\partial_x V(x)$$

$$\therefore \quad [\, \partial_x , f(x) \,] \, V(x) \;=\; V(x)\partial_x f(x) \tag{13.2}$$

$$[\, \partial_x , f(x) \,] \;=\; \partial_x f(x) \tag{13.3}$$

$$\Big\{ \, (V_c - V_d) - (V_b - V_a) \,] \;-\; [\, (V_c - V_b) - (V_d - V_{a\prime}) \, \Big\} = V_a - V_{a\prime}$$

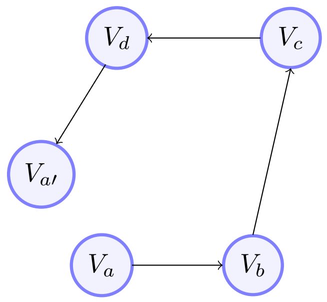
The First term of the first term is related to $dV = \dfrac{\partial V}{\partial x^\mu} dx^\mu$

while the first term of the second term is related to $-dV = -\dfrac{\partial V}{\partial x^\nu} dx^\nu$

or more specifically, the oridinary partial derivative is replaced by covarient derivative,

The First term is related to $\quad dV \;=\; dx^\mu \, dx^\nu \, \nabla_\nu \, \nabla_\mu V$

while the Second term is related to $\; -\, dV \;=\; -\, dx^\mu dx^\nu \nabla_\mu \nabla_\nu V$



$$\delta V \;=\; dx^\mu dx^\nu \Big\{ \nabla_\nu \, \nabla_\mu - \nabla_\mu \, \nabla_\nu \Big\} V$$

$$\delta V \;=\; dx^\mu dx^\nu [\nabla_\nu \, , \, \nabla_\mu] \qquad (Refer \quad Equation\ 13.1) \tag{13.4}$$

13.2 Quest to find The Commutator of Covarient Derivatives

Using the Equation 8.16,

$$\begin{aligned}
[\, \nabla_\nu \, , \, \nabla_\mu \,] \;&=\; \nabla_\nu \, \nabla_\mu - \nabla_\mu \, \nabla_\nu \\
&=\; (\partial_\nu \;-\; \Gamma_\nu) \; (\partial_\mu \;-\; \Gamma_\mu) \;-\; (\partial_\mu \;-\; \Gamma_\mu) \; (\partial_\nu \;-\; \Gamma_\nu) \\
&=\; [\partial_\mu \; , \; \Gamma_\nu] \;-\; [\partial_\nu \; , \; \Gamma_\mu] \;+\; \Gamma_\nu \Gamma_\mu \;-\; \Gamma_\mu \Gamma_\nu \\
&=\; \partial_\mu \, \Gamma_\nu \;-\; \partial_\nu \, \Gamma_\mu \;+\; \Gamma_\nu \, \Gamma_\mu \;-\; \Gamma_\mu \, \Gamma_\nu \\
\mathbf{R}^\alpha{}_{\nu\mu\beta} &=\; \partial_\mu \, \Gamma^\alpha{}_{\nu\beta} \;-\; \partial_\nu \, \Gamma^\alpha{}_{\mu\beta} \;+\; \Gamma^\alpha{}_{\nu\delta} \, \Gamma^\delta{}_{\mu\beta} \;-\; \Gamma^\alpha{}_{\mu\delta} \, \Gamma^\delta{}_{\nu\beta}
\end{aligned}$$

$$R^\alpha{}_{\nu\mu\beta} = \partial_\mu \, \Gamma^\alpha{}_{\nu\beta} \;-\; \partial_\nu \, \Gamma^\alpha{}_{\mu\beta} \;+\; \Gamma^\alpha{}_{\nu\delta} \, \Gamma^\delta{}_{\mu\beta} \;-\; \Gamma^\alpha{}_{\mu\delta} \, \Gamma^\delta{}_{\nu\beta} \tag{13.5}$$

Here Equation 13.5 is the **Riemann Curvature Tensor**.

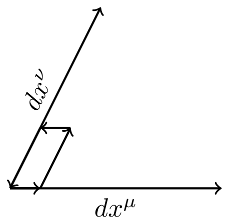
13.3 Riemann Tensor: Another Approach

In parallel transporting a quantity around a loop, the differnetial of the quantity will be proportional to the area of the loop,

For a 2-dimensional, 2+ space,

$$\delta V^{\alpha} \ = \ dx^{\mu} \ dx^{\nu} \ R^{\alpha}{}_{\mu\nu\beta} \ V^{\beta}$$

Here, $R^{\alpha}{}_{\mu\nu\beta}$ is the Riemann Tensor.



13.3.1 Insights:

- Two indices of the Riemann Tensor comes from the differential element of the space-time and the other two comes from the parallel transported vector.
- $R^{\alpha}{}_{\mu\nu\beta}$ is symmetric in μ, ν and α, β,

$$R^{\alpha}{}_{\mu\nu\beta} \ = \ R^{\beta}{}_{\mu\nu\alpha} \ = R^{\alpha}{}_{\nu\mu\beta}$$

- Lowering Index:

$$R_{\mu\nu\beta\alpha} \ = \ g_{\alpha\lambda} \ R_{\mu\nu\beta}{}^{\lambda}$$

- $R_{\alpha\beta\mu\nu}$,

$$R_{\alpha\beta\mu\nu} \ = \ - \ R_{\beta\alpha\mu\nu} \ = \ - \ R_{\alpha\beta\nu\mu} \ = \ R_{\beta\alpha\nu\mu}$$

Parallel Transport: If we move **ACW** Anti-Clockwise around the loop,

- If the Deflection (Deficit) is **ACW** $\longrightarrow$ **Positive (+ve)** Curvature
- If the Deflection (Deficit) is **CW** $\longrightarrow$ **Negative (-ve)** Curvature

13.4 Ricci Tensor

For $R^{\alpha}{}_{\mu\beta\nu}$ Ricci Tensor is defined as,

$$R_{\mu\nu} \ = \ R^{\alpha}{}_{\mu\alpha\nu} \tag{13.6}$$

Insights:

- Symmetry, can be defined using the Riemann Tensor's definations,

$$R_{\mu\nu} = R_{\nu\mu}$$

- $R_{\mu\nu} \ = \ 0$ is necessary but not sufficient condition for flat space.
- $R_{\alpha\beta\mu\nu} \ = \ 0$ is a necessary and sufficient condition for flat space.

13.5 Curvature Scaler: Ricci Scaler

$$R \ = \ g^{\mu\nu} \ R_{\mu\nu} \tag{13.7}$$

It can be defined as the trace of the Ricci Tensor.

$R \ = \ 0$, it's necessary but not sufficient condition for flat space.